# NORTH SYDNEY BOYS HIGH SCHOOL

## 2021 YEAR 12 HSC ASSESSMENT TASK 3

# Mathematics Extension 1

Student Number: ______________________________

### General Instructions

- **Reading time – 10 minutes**
- **Working time – 2 hours**
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Write Multiple Choice responses on sheet provided
- Write on the lined paper in the booklets provided
- Use the Reference Sheet

**Section I – 10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

**Section II – 60 marks**

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

**Class Teacher:**
(Shade the circle)

- O Mr Berry
- O Mr Lin
- O Mr Umakanthan
- O Mr Ireland
- O Ms Lee
- O Dr Vranesevic

| Question No | 1-10 | 11 | 12 | 13 | 14 | Total | % |
|---|---|---|---|---|---|---|---|
| Mark | $\overline{10}$ | $\overline{15}$ | $\overline{15}$ | $\overline{15}$ | $\overline{15}$ | $\overline{70}$ | $\overline{100}$ |

| 2021 Y12 Ext 1 – Assessment Task 3 - Outcomes | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Question | ME11-1 | ME11-2 | ME11-3 | ME11-4 | ME12-1 | ME12-2 | ME12-3 | ME12-4 | |
| 1 | | | | | /1 | | | | |
| 2 | | /1 | | | | | | | |
| 3 | | | /1 | | | | | | |
| 4 | | | | | | /1 | | | |
| 5 | | /1 | | | | | | | |
| 6 | | | | /1 | | | | | |
| 7 | /1 | | | | | | | | |
| 8 | /1 | | | | | | | | |
| 9 | | | | | | /1 | | | |
| 10 | | | | | /1 | | | | |
| 11(a) | /1 | | | | | | | | |
| 11(b) | | | | | | /2 | | | |
| 11(c) | | | /2 | | | | | | |
| 11(d) | | | | | /3 | | | | |
| 11(e) | | | | | | | /7 | | |
| 12(a) | | | | | /3 | | | | |
| 12(b) | | | | | | | /4 | | |
| 12(c) | | /5 | | | | | | | |
| 12(d) | | | | | | /3 | | | |
| 13(a) | | | | | /4 | | | | |
| 13(b) | | | | | | | /2 | | |
| 13(c) | | | | | | | /3 | | |
| 13(d) | | | | | /3 | | | | |
| 13(e) | | | | /3 | | | | | |
| 14(a) | | | | | | /2 | | | |
| 14(b) | | | | | | | | /4 | |
| 14(c) | | | | | /2 | | | | |
| 14(d) | | | | | | /5 | | | |
| 14(e) | | | | | /2 | | | | |
| Total | /3 | /7 | /3 | /4 | /19 | /14 | /16 | /4 | /70 |

**Section I**
**10 marks**
**Attempt Questions 1-10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1-10

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1. Find $\int \frac{4}{25+16x^2}\,dx$

(A) $\frac{1}{5}\tan^{-1}\frac{5x}{4}+C$

(B) $\frac{1}{5}\tan^{-1}\frac{4x}{5}+C$

(C) $5\tan^{-1}\frac{5x}{4}+C$

(D) $5\tan^{-1}\frac{4x}{5}+C$

2. Find the value of $a$ such that $P(x)=x^3-2x^2-ax+6$ is divisible by $x+2$.

(A) $-5$

(B) $-3$

(C) $3$

(D) $5$

3. The domain and range of the function $f(x)=4\sin^{-1}(3x+1)$ are respectively

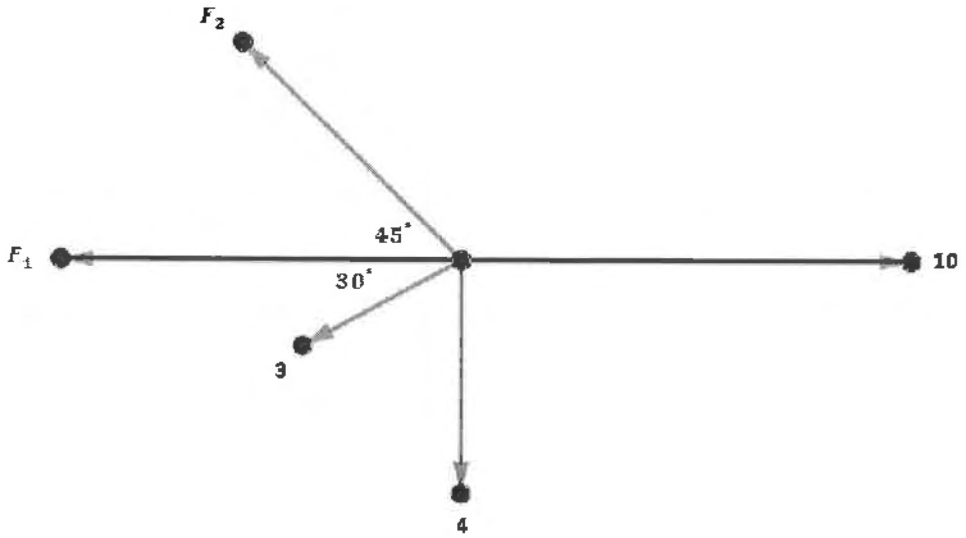
(A) $0\le x\le \frac{2}{3}$ and $-2\pi\le y\le 2\pi$

(B) $-\frac{2}{3}\le x\le 0$ and $-2\pi\le y\le 2\pi$

(C) $-\frac{2}{3}\le x\le 0$ and $\pi\le y\le 2\pi$

(D) $0\le x\le \frac{2}{3}$ and $\pi\le y\le 2\pi$

4. The diagram below shows a mass being acted upon by a number of forces whose magnitudes are labelled. All forces are in newtons, and the system is in equilibrium.




The value of $F_2$ is

(A) $\frac{\sqrt{2}}{2}(8 + 3\sqrt{3})$

(B) $\frac{11\sqrt{2}}{2}$

(C) $\frac{3\sqrt{2}}{2}$

(D) 7

5. Let $\alpha, \beta, \gamma$ be the zeroes of $x^3 - 4x + 1 = 0$. What is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ ?

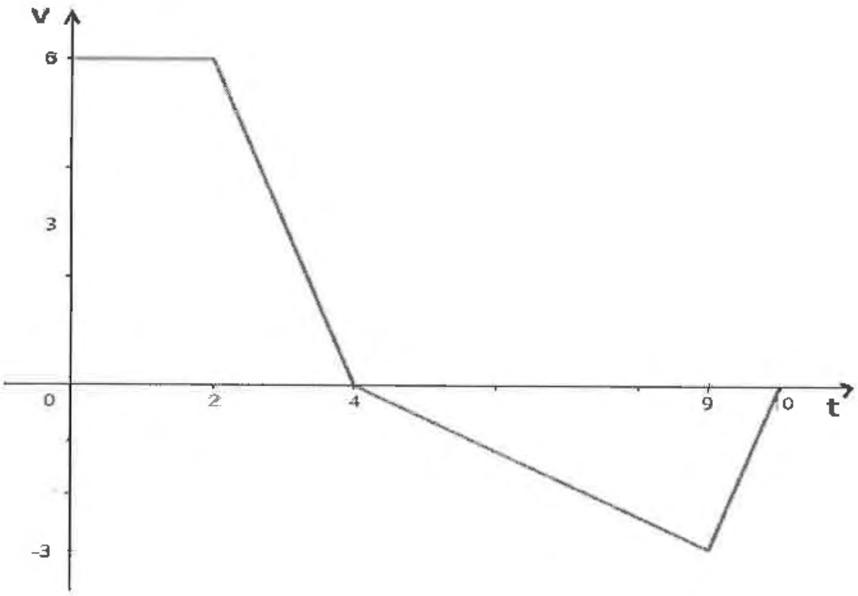
(A) $-4$

(B) $-1$

(C) 1

(D) 4

**6.** The diagram below shows the velocity-time graph of an object that moves over a 10 second interval. For what percentage of the time is the speed of the object decreasing?




(A) 30 %

(B) 60 %

(C) 70 %

(D) cannot tell from the graph

7. The function $f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function with which restriction of its domain ?

(A) $D = (7, \infty)$

(B) $D = (-4, 8)$

(C) $D = (-\infty, 0)$

(D) $D = [-\frac{1}{2}, \infty)$

8. Given $f(x) = \frac{3}{x} - 4$, $f^{-1}(4)$ equals

(A) $-\frac{13}{4}$

(B) $\frac{13}{4}$

(C) $\frac{3}{8}$

(D) $-\frac{3}{8}$

9. Consider the vectors given by $\underline{a} = m\underline{i} + \underline{j}$ and $\underline{b} = \underline{i} + m\underline{j}$, where $m \in \mathbb{R}$.

If the acute angle between $\underline{a}$ and $\underline{b}$ is $30°$, then $m$ equals

(A) $\sqrt{2} \pm 1$

(B) $2 \pm \sqrt{3}$

(C) $\sqrt{3}, \frac{1}{\sqrt{3}}$

(D) $\frac{\sqrt{3}}{4-\sqrt{3}}$

10. Evaluate the integral $\int_0^{\frac{1}{2}} \frac{sin^{-1}x}{\sqrt{1-x^2}}\, dx$

(A) $\frac{\pi^2}{72}$

(B) $\ln 6 - \ln \pi$

(C) $\frac{\pi^2}{36}$

(D) $\frac{\pi}{\sqrt{2}}$

# Section II

**60 Marks**

**Attempt Questions 11-14**

**Allow about 1 hour 45 minutes for this section.**

Answer each question in the appropriate writing booklet.
Extra writing booklets are available.

In Questions 11 – **14**, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use the Question 11 writing booklet

(a) Find the Cartesian equation represented by $x = \cos t + \sin t$ and $y = \cos t - sin\ t$ **1**

(b) Given the vectors $\underset{\sim}{c} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{d} = 9\underset{\sim}{i} + \underset{\sim}{j}$, find the projection of $\underset{\sim}{c}$ onto $\underset{\sim}{d}$. **2**

(c) Evaluate $tan^{-1}(\sqrt{3}) - tan^{-1}(-1)$ **2**

(d) Find the coefficient of $x^8$ in the expansion of $(\frac{x}{4} - \frac{4}{x})^{14}$ **3**

(e)

(i) Find all solutions for $\frac{x}{1-x^2} \geq 0$ **3**

(ii) Show that $\sec x \tan x = \frac{sin\, x}{1-sin^2 x}$ **1**

(iii) Hence or otherwise find all solutions of $\sec x \tan x \geq 0$, $0 \leq x \leq 2\pi$ **3**

*Examination continues on the next page* ......................................................

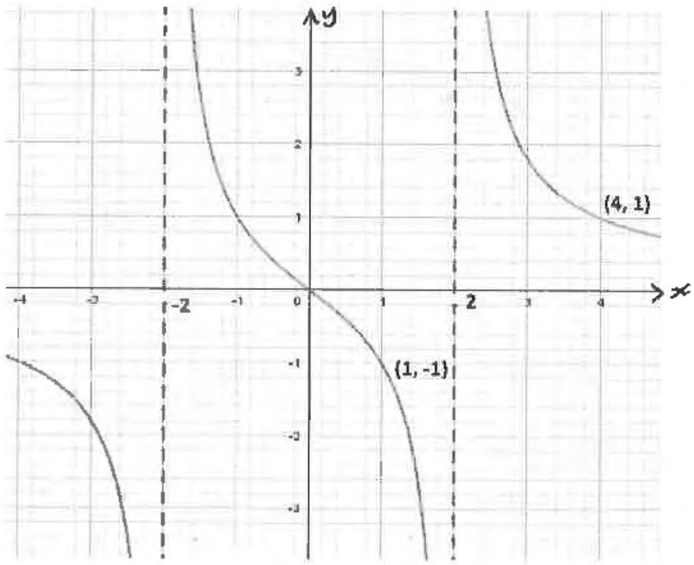
**Question 12** (15 marks) Use the Question 12 writing booklet

(a) The polynomial $P(x) = x^4 - x^3 - 11x^2 + 9x + 18$ has four distinct zeroes. Find them. **3**

(b) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$ where $R > 0$ and $0 \leq \alpha \leq 2\pi$. **2**

(ii) Hence solve the inequality $\sin x + \sqrt{3}\cos x \geq 1$ in the domain $[0, 2\pi]$. **2**

(c) The diagram below shows the graph of the odd function $y = f(x)$ where $f(x) = \frac{3x}{x^2-4}$

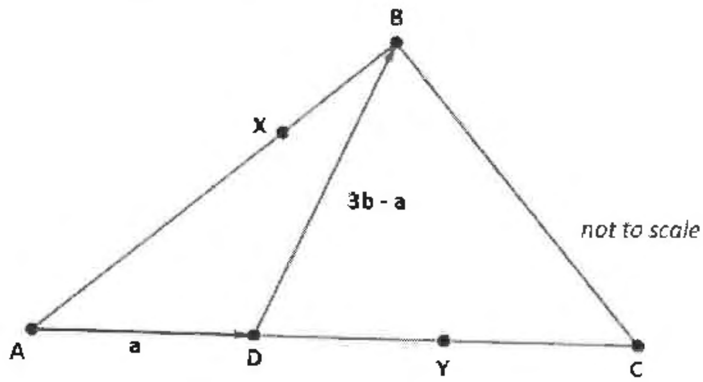



Sketch the graphs of the following functions in separate large diagrams, showing $x$-intercepts and asymptotes (stationary points are not needed):

(i) $y = f(|x|)$ **2**

(ii) $y = \frac{1}{f(x)}$ **3**

(d) On the diagram below, $\overrightarrow{AD} = a$, $\overrightarrow{DB} = 3b - a$ and $AC$ and $AB$ are straight lines.
You are given that $AD : DY : YC = 1 : 1 : 1$ and $AX : XB = 2 : 1$




Show that $XY$ is parallel to $BC$. **3**

*Examination continues on the next page* ......................................................

**Question 13** (15 marks) Use the Question 13 writing booklet

(a) Evaluate the integral $\int_0^{\sqrt{5}} \frac{2x^3}{\sqrt{x^2+4}}\, dx$ using the substitution $u = x^2 + 4$. **4**

(b) Find $\int \frac{2\tan 5\theta}{sec^2 5\theta}\, d\theta$ **2**

(c) Using sums to products of trigonometric ratios, solve the equation

$\cos x + \cos 2x + \cos 3x = 0, \quad 0 \leq x \leq 2\pi$ **3**

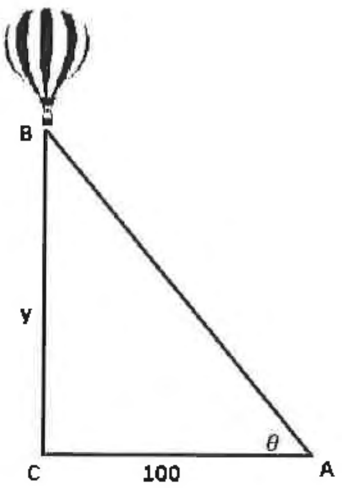
(d) Prove by induction that for all integers $n \geq 1$,

$\sum_{r=1}^{n} r \times 2^{r-1} = (n-1) \times 2^n + 1$ **3**

(e) An observer at A is watching balloon B as it rises from the ground at point C.

The balloon is rising at a constant rate of 3 metres per second and the observer is 100 metres from point C. The angle of elevation of the balloon is $\theta$ radians.

Find the rate of change of $\theta$ with respect to time at the instant the balloon is 50 metres off the ground.




**3**

*Examination continues on the next page* ......................................................

**Question 14** (15 marks) Use the Question 14 writing booklet

(a) The vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are such that $|\underset{\sim}{u}| = 4$ and $|\underset{\sim}{v}| = 5$.

If $\underset{\sim}{u} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = 21$, determine the size of the angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

(b) The region enclosed by $y = \sin x$ and $y = \sin 2x$ in the domain $0 \le x \le \frac{\pi}{2}$ is rotated about the $x$-axis. Calculate the volume of the resulting solid of revolution. **4**

(c) Consider the function $f(x) = x^3 + 2x$ and its inverse function $g(x)$.

Use the derivative of $f[g(x)]$ to find the gradient of the tangent to $y = g(x)$ at the point where $x = 3$. **2**

(d) A ball is thrown uphill from the base of a hill inclined at 45 degrees to the horizontal. The initial velocity was 20 metres per second at 60 degrees to the horizontal.

The position vector of the centre of the ball at t seconds after release is given by

$\underset{\sim}{s}(t) = 10t\,\underset{\sim}{i} + \left(10\,t\sqrt{3} - 5t^2\right)\underset{\sim}{j}$ where $\underset{\sim}{i}$ is a unit vector in the horizontal direction of motion of the ball, and $\underset{\sim}{j}$ is a unit vector vertically upward.

(i) Find the Cartesian equation of the ball's path. **1**

(ii) Find its time of flight. **2**

(iii) Calculate the ball's speed when it strikes the hill. **2**

(e) The function $y = f(x)$ is odd, and $f(x) \ge 0$ for $x \ge 0$.

$\int_0^a f(x)\,dx = e^a + \frac{1}{e^a} - 2$ if $a > 0$.

Find the area bounded by $y = f(x + 4)$ and the straight lines $x = 5, x = -5$, and the $x$-axis. (Leave answer in exact form). **2**

**END OF EXAMINATION**

Q1. B

Q2. D

Q3. B

Q4. B

Q5. D

Q6. A

Q7. A

Q8. C

Q.9. C

Q10. A

Q11

(a) $\begin{cases} x = \cos t + \sin t \\ y = \cos t - \sin t \end{cases}$

$\therefore x^2+y^2 = \cos^2 t + \cancel{2\cos t \sin t} + \sin^2 t$
$+\cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t$
$= 2\cos^2 t + 2\sin^2 t$

$\therefore x^2+y^2 = 2$ ✓

(b) $\underset{\sim}{c} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ , $\underset{\sim}{d} = 9\underset{\sim}{i} + \underset{\sim}{j}$

$\therefore \text{proj}_{\underset{\sim}{d}}\, \underset{\sim}{c} = \dfrac{\underset{\sim}{c}\cdot\underset{\sim}{d}}{\underset{\sim}{d}\cdot\underset{\sim}{d}} \times \underset{\sim}{d}$

$= \dfrac{18+3}{81+1}(9\underset{\sim}{i} + \underset{\sim}{j})$ ✓

$= \dfrac{21}{82}(9\underset{\sim}{i} + \underset{\sim}{j})$ ✓

(c) $\tan^{-1}\sqrt{3} - \tan^{-1}(-1) = \dfrac{\pi}{3} - \left(-\dfrac{\pi}{4}\right)$ ✓

$= \dfrac{7\pi}{12}$ ✓

(d) $\left(\dfrac{x}{4} - \dfrac{4}{x}\right)^{14} = \displaystyle\sum_{k=0}^{14} {}^{14}C_k \left(\dfrac{x}{4}\right)^{14-k}\left(-\dfrac{4}{x}\right)^k$

$= \displaystyle\sum_{k=0}^{14} {}^{14}C_k\, x^{14-k}\cdot 4^{k-14}\cdot(-1)^k\, 4^k\, x^{-k}$

$= \displaystyle\sum_{k=0}^{14} {}^{14}C_k\, x^{14-2k}\, (-1)^k\, 4^{2k-14}.$ ✓ working

we want $x^8$ $\therefore 14-2k = 8$ $\therefore$ $k=3$ ✓ $k=3$

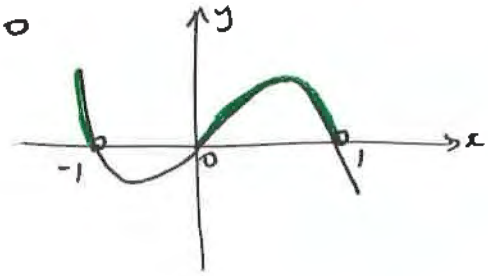
$\therefore$ coeff. $= {}^{14}C_3(-1)^3\, 4^{-8} = \dfrac{-{}^{14}C_3}{4^8} = -\dfrac{91}{16384}$ ✓

Q11 - ctd

(d) (i) $\dfrac{x}{1-x^2} \geq 0 \qquad \therefore x \neq -1, 1$

$\therefore \quad x(1-x^2) \geq 0 \qquad$ (mult. by $(1-x^2)^2$) ✓

$x(1-x)(1+x) \geq 0$




$\therefore \quad x < -1 \quad$ or $\quad 0 \leq x < 1$ ✓✓

(ii) $\sec x \tan x = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x}$

$= \dfrac{\sin x}{\cos^2 x}$

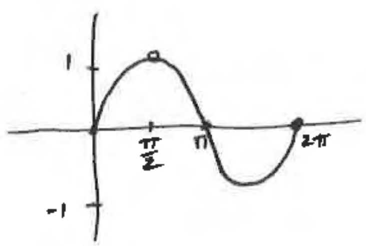
$= \dfrac{\sin x}{1-\sin^2 x}$ , as required. ✓

(iii) $\sec x \tan x \geq 0 \; , \; 0 \leq x \leq 2\pi$

gives $\dfrac{\sin x}{1-\sin^2 x} \geq 0$.

$\therefore$ from (i), $\sin x < -1 \quad$ or $\quad 0 \leq \sin x < 1$ ✓

no solution




$\therefore \; 0 \leq x < \frac{\pi}{2}$

or

$\frac{\pi}{2} < x < \pi$

or

$x = 2\pi$ ✓✓

Q12 $P(x) = x^4 - x^3 - 11x^2 + 9x + 18$

(a)

$P(0) = 18 \neq 0$

$P(1) = 1 - 1 - 11 + 9 + 18 \neq 0$

$P(-1) = 1 + 1 - 11 - 9 + 18 = 0 \quad \therefore -1$ is a zero

$P(2) = 16 - 8 - 44 + 18 + 18 = 0 \quad \therefore 2$ is a zero. ✓ Finds one zero

Let other zeros be $\alpha, \beta$.

By sum of zeros, $\alpha + \beta + (-1) + 2 = 1$

$\therefore \alpha + \beta = 0 \quad \text{---- (1)}$

By product of zeros, $\alpha\beta(-1)(2) = 18$

$\therefore \alpha\beta = -9 \quad \text{--- (2)}$

By inspection, or otherwise, $\alpha = 3$, $\beta = -3$ (or reverse!)

$\therefore$ the zeros are $-1, 2, 3, -3$ ✓✓

(b) (i) Let $\sin x + \sqrt{3}\cos x = R\sin(x + \alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

equate like terms:

$R\cos\alpha = 1$

$R\sin\alpha = \sqrt{3}$

Square & add $\rightarrow$ $R^2 = 1 + 3 = 4 \quad \therefore R = 2$ (as $R > 0$)

Also, $\tan\alpha = \sqrt{3} \quad \therefore \alpha = \frac{\pi}{3}$ ✓✓

$\therefore = 2\sin\left(x + \frac{\pi}{3}\right)$

[max. 1 mark if incomplete working].

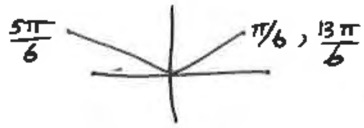
Q12 - b - cont$^d$.

(ii) $\sin x + \sqrt{3}\cos x \geq 1$ , $0 \leq x \leq 2\pi$

gives $2 \sin\left(x + \frac{\pi}{3}\right) \geq 1$

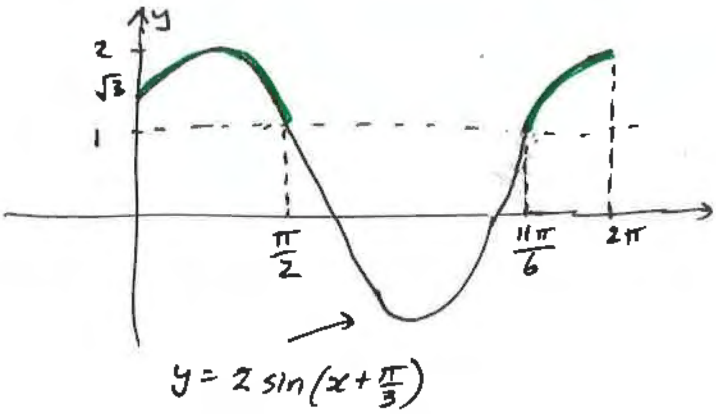
$\sin\left(x + \frac{\pi}{3}\right) \geq \frac{1}{2}$ , $0 \leq x \leq 2\pi$.

Let's solve $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ , $\frac{\pi}{3} \leq x + \frac{\pi}{3} \leq \frac{7\pi}{3}$




$\therefore$ $x + \frac{\pi}{3} = \frac{5\pi}{6}, \frac{13\pi}{6}$

$\therefore$ $x = \frac{\pi}{2}, \frac{11\pi}{6}$ ✓

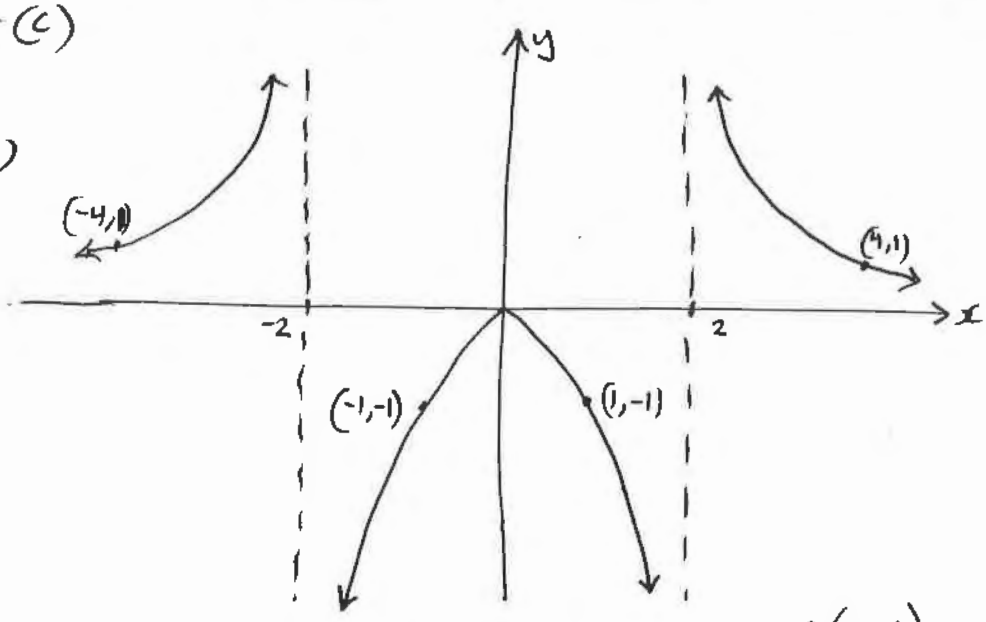



$\therefore$ Solution is $0 \leq x \leq \frac{\pi}{2}$ or $\frac{11\pi}{6} \leq x \leq 2\pi$ ✓

i.e. $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$

Q 12 - (c)

(i)




$y = f(|x|)$

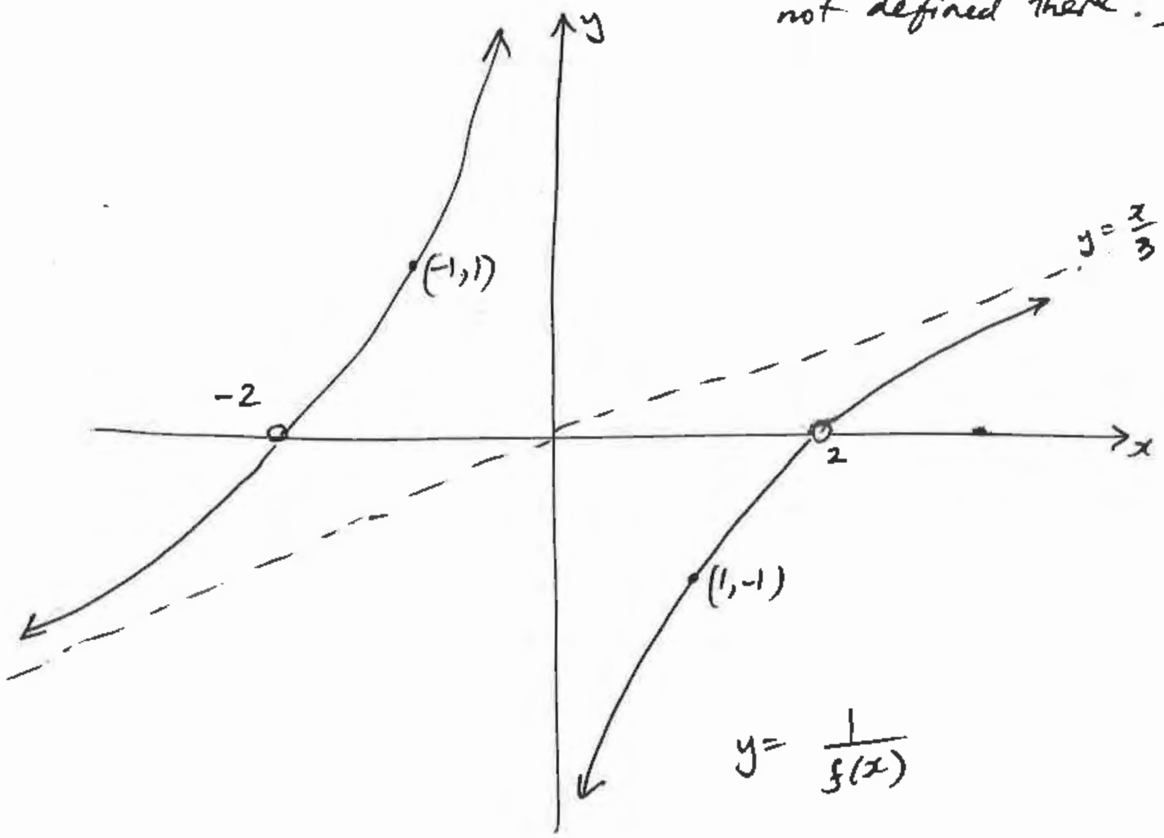
✓✓

Shape, asymptotes, some ID points.

(ii) $y = \frac{1}{f(x)} = \frac{x^2-4}{3x} = \frac{x}{3} - \frac{4}{3x} \quad (x \neq -2, 2)$

$\therefore$ oblique asymptote is $y = \frac{x}{3}$.

[Also, when $x = \pm 2$, curve passes thru $(-2,0)$ & $(2,0)$, but is not defined there.]

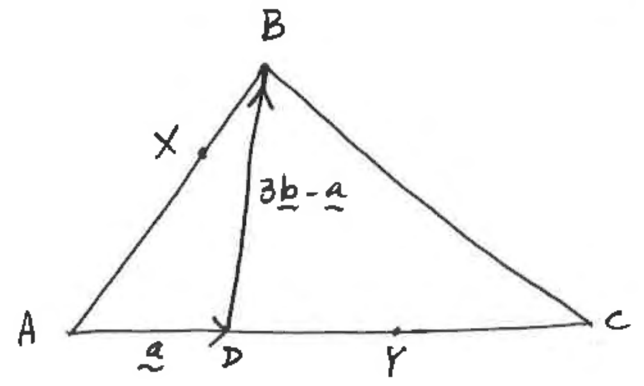



$y = \frac{1}{f(x)}$

✓✓✓

Needs open circles and oblique asymptote for full marks.

Q12 (d)




Given: $AD : DY : YC = 1:1:1$
$AX : XB = 2:1$

To prove: $XY \parallel BC$

$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{BD} = \underset{\sim}{a} + (3\underset{\sim}{b} - \underset{\sim}{a}) = 3\underset{\sim}{b}$

$\therefore \quad \overrightarrow{AX} = \frac{2}{3}\overrightarrow{AB} = 2\underset{\sim}{b}$ ✓ for $\overrightarrow{AX}$

Thus $\overrightarrow{XY} = \overrightarrow{AY} - \overrightarrow{AX}$

$= 2\underset{\sim}{a} - 2\underset{\sim}{b}$ ✓ for $\overrightarrow{XY}$

Also, $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$= 3\underset{\sim}{a} - 3\underset{\sim}{b}$

$\therefore \quad \overrightarrow{BC} = \frac{3}{2}\overrightarrow{XY}$

$= k\,\overrightarrow{XY}$, $k$ a constant.

i.e. $BC \parallel XY$.

✓ for $\overrightarrow{BC}$ & conclusion

Q13

(a) $I = \displaystyle\int_0^{\sqrt{5}} \frac{2x^3}{\sqrt{x^2+4}}\,dx \qquad [u = x^2+4]$

$\frac{du}{dx} = 2x \quad \therefore du = 2x\,dx.$

Also, $x^2 = u - 4$

when $\begin{cases} x=0, & u = 4 \\ x=\sqrt{5}, & u = 9. \end{cases}$ ✓ preparatory work.

$\therefore \quad I = \displaystyle\int_4^9 \frac{u-4}{\sqrt{u}}\,du$ ✓

$= \displaystyle\int_4^9 \left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right) du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right]_4^9$ ✓

$= \left[\frac{2}{3}(27) - 8(3)\right] - \left[\frac{2}{3}(8) - 8(2)\right]$

$= 18 - 24 - \frac{16}{3} + 16$

$= \frac{14}{3}$ ✓

Q13 -ctd

(b) $I = \int \frac{2\tan 5\theta}{\sec^2 5\theta}\, d\theta$

Method 1: $I = \int 2\frac{\sin 5\theta}{\cos 5\theta} \cdot \cos^2 5\theta \, d\theta$

$= \int 2\sin 5\theta \cos 5\theta \, d\theta$ ✓ progress

$= \int \sin 10\theta \, d\theta$ ✓ answer

$= -\frac{1}{10}\cos 10\theta + C$

Method 2: $I = \int \frac{2\tan 5\theta}{1+\tan^2 5\theta}\, d\theta$

$= \int \sin 10\theta \, d\theta$ (t results) (✓)

$= -\frac{1}{10}\cos 10\theta + C$ (✓)

(Note: $I = \frac{\sin^2 5\theta}{2} + C_2$ is also correct.)

(c) $\cos x + \cos 2x + \cos 3x = 0, \quad 0 \le x \le 2\pi$

We use $2\cos A\cos B = \cos(A-B) + \cos(A+B)$

Here, let $A = 2x$, $B = x$ $\therefore \cos(2x-x) + \cos(2x+x)$
$= 2\cos 2x\cos x$.

Thus $\cos 2x + 2\cos 2x\cos x = 0$ ✓

$\cos 2x\,[1 + 2\cos x] = 0$ $\left(\begin{array}{l} 0 \le x \le 2\pi \\ \therefore 0 \le 2x \le 4\pi \end{array}\right)$

$\cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Thus $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}$ ✓✓

Q13 – ctd

(d) To prove: $\sum_{r=1}^{n} r \times 2^{r-1} = (n-1)\,2^n + 1 \quad (n \geq 1)$

When $n = 1$

$LHS = 1 \times 2^{1-1}$
$= 1 \times 2^0$
$= 1$

$RHS = (1-1)\cdot 2^1 + 1$
$= 1$
$= LHS.$

$\therefore$ true for $n = 1$

✓ base case.

Assume true for $n = k$

ie. assume $1 \times 2^{1-1} + 2 \times 2^{2-1} + \dots + k \times 2^{k-1} = (k-1)\cdot 2^k + 1$ $(*)$

Then, when $n = k+1$,

$1 \times 2^{1-1} + 2 \times 2^{2-1} + \dots + k \times 2^{k-1} + (k+1) \times 2^{(k+1)-1}$

$= (k-1)\cdot 2^k + 1 + (k+1)\cdot 2^{(k+1)-1}$ by $(*)$

✓ uses assumption

$\therefore$

$LHS = (k-1)\cdot 2^k + 1 + (k+1)\cdot 2^k$

$= k\cdot 2^k - \cancel{2^k} + 1 + k\cdot 2^k + \cancel{2^k}$

$= 2 \times k \cdot 2^k + 1$

$= k \cdot 2^{k+1} + 1$

$= [(k+1) - 1]\cdot 2^{k+1} + 1$

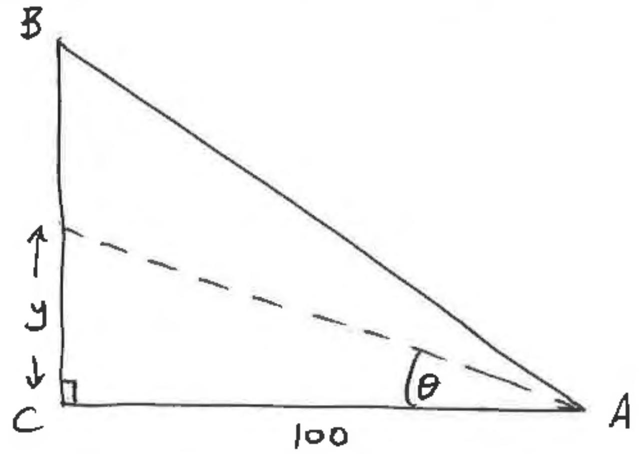
$\therefore$ if true for $n = k$, it's true for $n = k+1$

✓ Shows with no fudges!

$\therefore$ Since true for $n = 1$, it's true by induction for all $n \geq 1$.

Q13 - ctd

(e)



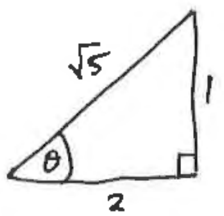

Let $y$ = height in m above ground at time $t$ secs.

Then $\frac{dy}{dt} = 3$ m/s (given).

$y = 100\tan\theta$

$\frac{dy}{d\theta} = 100\sec^2\theta$ ✓

When $y = 50$m, $\tan\theta = \frac{1}{2}$




$\therefore \sec\theta = \frac{\sqrt{5}}{2}$

$\therefore \frac{dy}{d\theta} = 100 \cdot \frac{5}{4} = 125$ ✓

Thus $\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$

$3 = 125 \cdot \frac{d\theta}{dt}$

$$\left.\begin{aligned}\frac{d\theta}{dt} &= \frac{3}{125} \text{ radians/s} \\ &= 0{\cdot}024 \text{ rads/s.}\end{aligned}\right\}$$ ✓ answer

Q14

(a) $|\underset{\sim}{u}| = 4, \quad |\underset{\sim}{v}| = 5$

$\underset{\sim}{u} \cdot (\underset{\sim}{u} \cdot \underset{\sim}{v}) = 21$

$\therefore \underset{\sim}{u} \cdot \underset{\sim}{u} + \underset{\sim}{u} \cdot \underset{\sim}{v} = 21$

$16 + \underset{\sim}{u} \cdot \underset{\sim}{v} = 21$

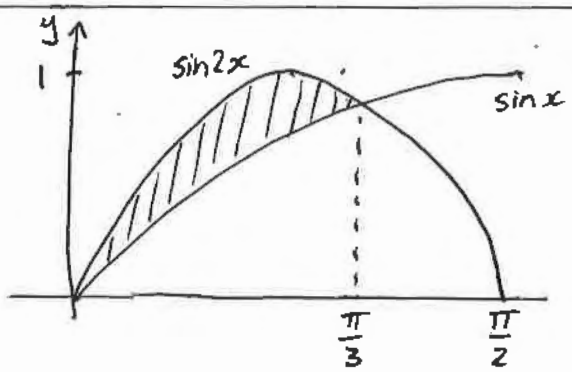
$\therefore \underset{\sim}{u} \cdot \underset{\sim}{v} = 5$ ✓

But $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}| \, |\underset{\sim}{v}| \cos\theta$

$\therefore \cos\theta = \dfrac{5}{4 \times 5} = \dfrac{1}{4}$

$\theta = 75^\circ 31'$ (nearest minute) ✓

(b)




$\sin 2x = \sin x$

$\therefore 2\sin x \cos x = \sin x$

$\sin x (2\cos x - 1) = 0$

$\therefore x = 0$ or $\frac{\pi}{3}$

$\therefore V = \pi \int_0^{\frac{\pi}{3}} (\sin^2 2x - \sin^2 x)\, dx$ ✓

$= \pi \int_0^{\frac{\pi}{3}} \frac{1}{2}(1 - \cos 4x) - \frac{1}{2}(1 - \cos 2x)\, dx$ ✓

$= \frac{\pi}{2} \int_0^{\frac{\pi}{3}} (\cos 2x - \cos 4x)\, dx$

$\therefore V = \frac{\pi}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{3}}$ ✓

$= \frac{\pi}{2}\left[\frac{\sqrt{3}}{4} - \left(-\frac{\sqrt{3}}{8}\right)\right] = \frac{\pi}{2}\left[\frac{3\sqrt{3}}{8}\right]$

i.e. $V = \dfrac{3\sqrt{3}\,\pi}{16}$ units$^3$. ✓

Q14 - (c)

$$f(x) = x^3 + 2x$$

$$g(x) = f^{-1}(x)$$

$$\therefore f[g(x)] = x$$

$$\therefore \frac{d}{dx} f[g(x)] = \frac{d}{dx} x$$

$$\therefore f'[g(x)] \cdot g'(x) = 1$$

$$g'(x) = \frac{1}{f'[g(x)]}.$$

we need $g(3)$:

If $3 = x^3 + 2x$ then, by inspection,

$$x = 1$$

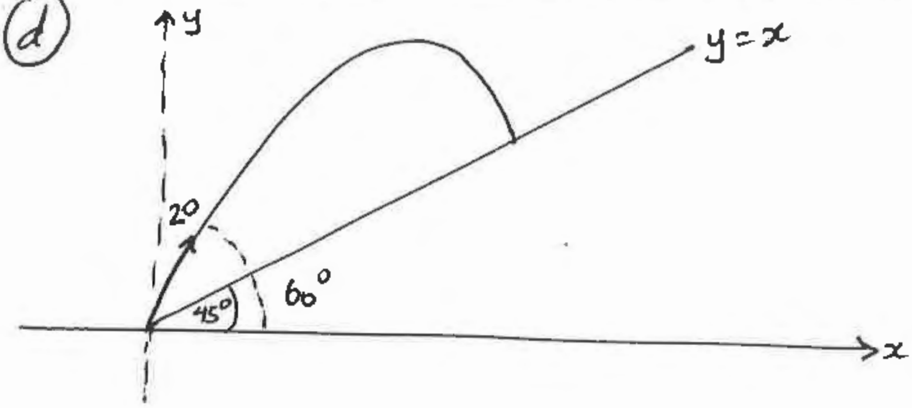
i.e. $g(3) = 1$

Since $f'(x) = 3x^2 + 2$

$$\therefore f'[g(3)] = f'(1)$$
$$= 3+2$$
$$= 5$$

$$\therefore g'(3) = \frac{1}{5}$$ ✓✓

Q14 - (d)




(i) $x = 10t \quad \therefore t = \frac{x}{10}$

$\therefore y = 10\sqrt{3}\left(\frac{x}{10}\right) - 5\left(\frac{x}{10}\right)^2$

$\therefore y = \sqrt{3}x - \frac{x^2}{20}$ ✓

(ii) ball strikes hill when ball & hill same height,

$\therefore \quad x = \sqrt{3}x - \frac{x^2}{20}$

$\therefore \quad \frac{x^2}{20} + (1-\sqrt{3})x = 0$ ✓ gets a quadratic

$x\left[\frac{x}{20} + 1 - \sqrt{3}\right] = 0$

$\therefore \frac{x}{20} = \sqrt{3} - 1 \quad \therefore \quad x = 20(\sqrt{3}-1)$

Thus $t = \frac{x}{10} \quad \therefore \quad t = 2(\sqrt{3}-1)$ secs. ✓ solves for t

(iii) $\dot{x} = 10, \quad \dot{y} = 10\sqrt{3} - 10t$

$= 10\sqrt{3} - 10\left[2(\sqrt{3}-1)\right]$

$= 10\sqrt{3} - 20\sqrt{3} + 20$

$\therefore \dot{y} = 20 - 10\sqrt{3}$ ✓ $\dot{x}$ & $\dot{y}$

Thus speed $= \sqrt{10^2 + (20-10\sqrt{3})^2}$

$= \sqrt{10^2 + 10^2(2-\sqrt{3})^2}$

$= 10\sqrt{1 + 4 - 4\sqrt{3} + 3}$

$= 10\sqrt{8 - 4\sqrt{3}}$

$= 20\sqrt{2-\sqrt{3}}$ m/s i.e. $\doteqdot 10.35$ m/s. ✓ speed.

Q14 - e

$y = f(x)$ is odd, $f(x) \geq 0$ for $x \geq 0$.

$$\text{Area} = \int_{-5}^{5} f(x+4)\,dx$$

Let $u = x+4$
$du = dx$
$x=5 \rightarrow u = 9$
$x=-5 \rightarrow u = -1$

$$= \int_{-1}^{9} f(u)\,du$$

✓

$$= \int_{-1}^{0} f(u)\,du + \int_{0}^{9} f(u)\,du$$

$$= \int_{0}^{1} f(u)\,du + \int_{0}^{9} f(u)\,du$$

as $f(x)$ is odd

$$= \left(e^1 + \frac{1}{e^1} - 2\right) + \left(e^9 + \frac{1}{e^9} - 2\right)$$

$\therefore$ Area $= e + \dfrac{1}{e} + e^9 + \dfrac{1}{e^9} - 4$ units$^2$. #

✓